ABSTRACT

In this paper, single snapshot direction-of-arrival (DOA) estimation under mutual coupling (MC) is considered with arbitrary array structures. A compressed sensing approach is utilized and a joint-sparse recovery algorithm is proposed. In this respect, both spatial source directions and MC coefficients are embedded into a joint-sparse vector. A new dictionary matrix is defined using the symmetry of MC matrix. The proposed approach does not depend on the structure of MC matrix and it is suitable for any type of array geometry. In the simulations, the proposed method is evaluated for a planar array where the antennas are placed randomly in 2-D space. It is shown that the proposed method effectively estimates the source parameters and performs significantly better than the alternative methods.

Index Terms—DOA estimation, Sparse recovery, Compressed sensing, Mutual coupling, Arbitrary array structures.

1. INTRODUCTION

Estimation of the direction-of-arrival (DOA) of unknown source locations is an important problem in mobile communications. While there are several high-resolution methods for this purpose such as the MUSIC algorithm [1], these algorithms fail if the array manifold is not perfectly known due to mutual coupling (MC) between the antennas.

The effect of MC is usually mitigated by using the special structure of the MC matrix for fixed array geometries such as linear, circular [2] or rectangular [3] arrays. Uniform linear array (ULA) has a banded Toeplitz MC matrix structure while uniform circular arrays (UCA) and uniform rectangular arrays (URA) have circulant [2] and Toeplitz-block [3] MC matrix structures respectively. An array with arbitrary geometry does not have any of these forms except that the MC matrix is symmetric [4]. Hence the distortion due to MC should be handled without using the simplicities of structured matrices for such arrays.

There are several methods proposed in the literature for DOA estimation under unknown MC [5–9]. These methods have two major limitations. Firstly, most of these methods are based on the eigen-structure of the array covariance matrix which is assumed to be accurately estimated using large number of snapshots. When there is a single snapshot, above algorithms do not provide accurate results. Secondly, these methods are only applicable for certain array geometries such as ULA, UCA and URA etc.

DOA estimation with arbitrary array structures is considered in limited number of studies and the methods based on array interpolation [10], manifold separation [11] and higher order statistics [12] are proposed. However, the effect of MC is ignored in these studies. Moreover, source signals are assumed to be independent in these studies while this is not valid for finite length data especially when the number of snapshots is small. Single snapshot DOA estimation is considered in [13] where a sparse recovery [14] technique is used to estimate the source DOA angles without considering the effect of MC.

In this paper, single snapshot DOA estimation problem under unknown MC is considered for arbitrary array structures. A joint-sparse recovery (JSR) algorithm is proposed where the unknown source directions and coupling coefficients are inserted into a joint-sparse vector so that the sparsity and convexity of the problem is preserved. A new dictionary matrix is generated using the symmetry of the MC matrix due to the reciprocity of the interactions between the antennas. To our knowledge, this is the first work that treats the single snapshot DOA estimation problem in case of MC for arbitrary array structures.

2. ARRAY SIGNAL MODEL

In this study, \( K \) narrowband source signals impinging from far-field on \( M \)-element randomly placed array are considered where the sources and the array are in the same plane. The MC matrix is denoted by \( \mathbf{C} \in \mathbb{C}^{M \times M} \) which is direction independent and symmetric [2]. The array output for a single snapshot can be written as follows

\[
\mathbf{y} = \mathbf{C}\hat{\mathbf{s}} + \mathbf{w}
\]  

where \( \mathbf{w} \) is spatially and temporally white, zero mean Gaussian noise with variance \( \sigma_w^2 \). \( \hat{\mathbf{s}} = [s_1, s_2, \ldots, s_K]^T \) is a \( K \times 1 \) vector composed of source signal amplitudes. \( \hat{\mathbf{A}} \) is \( M \times K \) nominal array steering matrix and defined as

\[
\hat{\mathbf{A}} = [\mathbf{a}(\hat{\theta}_1), \mathbf{a}(\hat{\theta}_2), \ldots, \mathbf{a}(\hat{\theta}_K)]
\]  

where \( \hat{\theta}_k \) represents the DOA angle of the \( k^{th} \) source. The \( m^{th} \) element of the array steering vector \( \mathbf{a}(\hat{\theta}_k) \) is given as

\[
a_m(\hat{\theta}_k) = [e^{j2\pi \tau_{1,k}}, e^{j2\pi \tau_{2,k}}, \ldots, e^{j2\pi \tau_{M,k}}]
\]  

where \( \tau_{m,k} = \frac{x_m \cos(\hat{\theta}_k) + y_m \sin(\hat{\theta}_k)}{\lambda} \) for \( (x_m, y_m) \) is the position of the \( m^{th} \) antenna and \( \lambda \) is the wavelength. The form of MC matrix, \( \mathbf{C} \), for arbitrary array structure is given as

\[
\mathbf{C} = \begin{pmatrix}
1 & \hat{c}_{12} & \cdots & \hat{c}_{1M} \\
\hat{c}_{21} & 1 & \cdots & \hat{c}_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{c}_{M1} & \hat{c}_{M2} & \cdots & 1
\end{pmatrix}
\]
Due to symmetry, $\bar{c}_{ij} = \bar{c}_{ji}$ and the distinct MC coefficients are given in the following set

$$S_c = \{\bar{c}_{ij} : i, j = 1, \ldots, M, \ i \neq j\} \cup \{1\}. \ (5)$$

The number of distinct MC coefficients is $M = \frac{M^2 - M}{2} + 1$ including the diagonal unit value term. The effect of MC decreases as the distance between the antenna pairs increases. When this distance is sufficiently large ($d_m > \lambda$ for $m = 1, \ldots, M$) the effect of MC can be ignored [15]. In this paper, the coupling coefficients of the antenna pairs with inter-element distance larger than a wavelength can be ignored [15]. In this paper, the coupling coefficients of the antenna pairs with inter-element distance larger than a wavelength are assumed to be zero and only the coefficients of the antenna pairs with $d_m < \lambda$ are considered. In the rest of the paper, the notation $\{c_m\}_{m=1}^M$ will be used for MC coefficients where $\{c_m\}_{m=1}^M$ is the set of distinct coupling coefficients in $S_c$. In other words, $\{c_m\}_{m=1}^M$ is defined with a single index $m$ for simplicity whereas $S_c$ includes two indices.

The problem we consider in this paper can be expressed as follows. Given the array output $y$ for a single snapshot, the sensor positions $(x_m, y_m)$ for $m = 1, \ldots, M$ and the number of sources $K$, estimate the source DOA angles $(\theta_1)_{K=1}$ and the coupling coefficients $\{c_m\}_{m=1}^M$.

### 3. JOINT-SPARSE RECOVERY FOR A SINGLE SNAPSHOT WITH MUTUAL COUPLING

The signal model in (1) can be written in the compressed sensing (CS) context for noise-free case as

$$y = \mathbf{C} s$$

where $s$ is $N \times 1$ K-sparse vector and $\mathbf{A}$ is $M \times N$ dictionary matrix defined as $\mathbf{A} = [a(\theta_1), a(\theta_2), \ldots, a(\theta_N)]$. The dictionary resolution in azimuth plane is $[\theta_1, \theta_2, \ldots, \theta_M] = \Delta_\theta, i = 1, \ldots, N - 1$. In order to find the support set of $s$ and $\mathbf{C}$ with CS, the following minimization problem can be considered

$$(P_0) \quad \min_{s \in \mathbb{R}^N; \ c \in \mathbb{C}^M} \|s\|_0 \quad s.t. \ y = \mathbf{C} s. \quad (7)$$

Since the problem $P_0$ is non-linear due to $\mathbf{C} s$ which includes both of the unknowns $s$ and $\mathbf{C}$, it should be modified for an effective solution. In the following part, $P_0$ is converted to a JSR problem which can be written in linear form. We introduce a joint-sparse vector composed of both signal vector $s$ and MC coefficients. Furthermore, a new dictionary matrix is defined using the symmetricity of the MC matrix $\mathbf{C}$.

MC matrix, $\mathbf{C}$, can be written as

$$\mathbf{C} = \sum_{m=1}^{M} c_m \mathbf{I}_m \quad (8)$$

where $\mathbf{I}_m$ is $M \times M$ matrix whose $(i,j)^{th}$ entry is given as

$$\mathbf{I}_m(i,j) = \begin{cases} 1 & \text{if } \mathbf{C}(i,j) = c_m, \ m = 1, 2, \ldots, M \\ 0 & \text{otherwise.} \end{cases} \ (9)$$

Then, $M \times MN$ dictionary matrix $\mathbf{D}$ can be defined by stacking $\mathbf{A}$ as $\mathbf{D} = [\mathbf{I}_1 \mathbf{A}, \mathbf{I}_2 \mathbf{A}, \ldots, \mathbf{I}_M \mathbf{A}]$.

In noise-free case, $P_1$, can be written in the following form, i.e.

$$(P_1) \quad \min_{x \in \mathbb{C}^{MN}} \|s\|_0 \quad s.t. \ y = \mathbf{D} x. \quad (10)$$

where $x = [s^T, c_2 s^T, c_3 s^T, \ldots, c_M s^T]^T$. The first coupling coefficient $c_1 = 1$ is assumed without loss of generality [2]. If $P_1$ is solved, $K$-sparse signal, $s$, can be recovered as the first $N$ entries of the $MK$-sparse signal $x$. In this paper, $MK$-sparse signal $x$ is said to be $K$-joint sparse if $\|s\|_0 = K$ where $\|\cdot\|_0$ denotes the joint-sparsity which is defined as

$$\|s\|_0 = |\{i : x_i(m) \neq 0\}|, \ m = 1, \ldots, M \quad (11)$$

where $x_i(m)$ is the $i^{th}$ entry of the $m^{th}$ block of $x$.

When the observation $y$ is noisy, i.e., $y = \mathbf{C} s + w$ where the noise power is bounded by $\|w\|_2^2 \leq \epsilon^2$, the constrained CS problem $P_1$ can be formulated with an inequality constraint. Furthermore, mixed $l_2, l_1$-norm can be utilized to relax the minimization problem and obtain an effective solution. In this case, JSR problem can be given as

$$(P_2) \quad \min_{x \in \mathbb{C}^{MN}} \|s\|_2, \ 1 \ s.t. \ y - \mathbf{D} x \leq \epsilon^2 \quad (12)$$

which can be explicitly given below

$$\|s\|_2,1 = \sum_{i=1}^{N} \left( \sum_{m=1}^{M} |x_{M(m-1)+i}|^2 \right)^{1/2} \quad (13)$$

$P_2$ can be further modified to a more convenient form by removing the inequality constraint and the final form of the proposed JSR algorithm is given as

$$(P_3) \quad \min_{x \in \mathbb{C}^{MN}} \mu \|s\|_2,1 + \frac{1}{2} \|y - \mathbf{D} x\|_2^2 \quad (15)$$

where $\mu$ is the penalty term which determines the trade-off between $l_2, l_1$-normed terms in the problem. $P_3$ is a convex problem and can be solved effectively by using standard techniques. Once it is solved, $K$-sparse $N \times 1$ vector $s$ can be obtained as the first $N$ entry of the $MK$-sparse $MN \times 1$ vector $x$. Then we can find the source DOA angles that correspond to the non-zero entries of $s$ in $M \times N$ dictionary matrix $\mathbf{A}$.

### 4. ESTIMATION OF MC COEFFICIENTS

Once the convex problem in (15) is solved, the coupling coefficients can be found as $c_m = x_{(M(m-1)+i)}/s_i$ for $m = 1, \ldots, M$ and $i \in \mathbb{I}_k$ which is the set of indices of $S_c$. This is a suboptimum method for finding the coupling coefficients since there exist $K$ many solutions of $c_m$, for $i \in \mathbb{I}_k$. An alternative and better solution is to find $\hat{s}$ and $\bar{s}$ from (15), then obtain a least-square solution of $c = [c_1, \ldots, c_M]^T$ from a cost function. In this case, $c$ is found by minimizing the cost function $J(c) = \|y - \sum_{m=1}^{M} c_m \mathbf{I}_m \mathbf{A} \bar{s}\|_2^2$, i.e.

$$\|y - \sum_{m=1}^{M} c_m \mathbf{I}_m \mathbf{A} \bar{s}\|_2^2 = y^H y - y^H \sum_{m=1}^{M} c_m \mathbf{I}_m \mathbf{A} \bar{s}$$

$$- \sum_{m=2}^{M} c_m \mathbf{I}_m \mathbf{A}^H y - \sum_{m=1}^{M} \sum_{m=2}^{M} c_{m_1} c_{m_2} \mathbf{I}_m \mathbf{I}_{m_2} \mathbf{A}^H y.$$
If the derivative of $J(c)$ with respect to $c_{m2}$ is considered and equated to zero, the following expression is obtained,
\[
\sum_{m_1=1}^{\tilde{M}} \sum_{m_2=1}^{\tilde{M}} c_{m1} s^H A^H I_{m2}^H I_{m1} A \ddot{s} = \sum_{m_2=1}^{\tilde{M}} s^H A^H I_{m2}^H y. \tag{16}
\]

Above equation can be written as a linear set of equations, $A_{c} c = b_{c}$, where the elements of $A_{c} \in \mathbb{C}^{M \times M}$ and $b_{c} \in \mathbb{C}^{M}$ are known and expressed as
\[
A_{c}(m_2, m_1) = s^H A^H I_{m2}^H I_{m1} A \ddot{s}, \quad b_{c}(m_2) = s^H A^H I_{m2}^H y, \quad m_1, m_2 = 1, \ldots, \tilde{M}. \tag{17}
\]

Then, the MC coefficients can be found as $c = A_{c}^{-1} b_{c}$.

5. UNIQUENESS OF THE SOLUTION

In this section, the uniqueness of the solution for the proposed method is considered for both DOA angle and MC coefficient estimation. For MC coefficient estimation, (17) is well-posed and solution is unique if $\tilde{M} - 1 \leq M$ is satisfied. For DOA estimation, the problem in (10) is underdetermined in general, i.e., there are infinitely many solutions. However, in [17], it is shown that the solution is unique if $\ddot{x}$ is sparse. In the following, we show that the solution of $P_1$ is unique if the matrix $D$ obeys the restricted isometry property for joint-sparse case (JS-RIP) defined below.

**Definition 1.** The dictionary $D$ is said to obey JS-RIP with sparsity level $K$, if there exists $\delta_K \in [0, 1)$ for all joint-sparse $(MK$-sparse) signal $x \in \mathbb{C}^{MN}$ such that
\[
(1 - \delta_K)||x||_2^2 \leq ||Dx||_2^2 \leq (1 + \delta_K)||x||_2^2 \tag{18}
\]
holds for $\delta_K$ which can be found as
\[
\min_{\delta_K \in [0, 1)} \delta_K s.t. \ (1 - \delta_K)||x||_2^2 \leq ||Dx||_2^2 \leq (1 + \delta_K)||x||_2^2 \tag{19}
\]

Then we consider the uniqueness of the solution for the problem $P_1$ in the following theorem [18].

**Theorem 1.** Let $s$ and $\ddot{s}$ be the solution to problem $P_1$. If the matrix $D$ obeys JS-RIP with $\delta_{2K} < 1$, then the solution is unique.

**Proof:** Let $\dddot{x}$ be the solution to $P_1$ with
\[
\dddot{x} = [s^T, c_2 s^T, c_3 s^T, \ldots, c_{\tilde{M}} s^T]^T. \tag{20}
\]

Then, we can say that $||x||_{0,1} \leq ||\dddot{x}||_{0,1} \leq K$ since both $x$ and $\dddot{x}$ are the solution. Using the triangle inequality, the difference is bounded by $2K$ as
\[
||x - \dddot{x}||_{0,1} \leq 2K. \tag{21}
\]

Since, both $\dddot{x}$ and $x$ solves $P_1$ with equality constraint, then, $y = Dx = D \dddot{x}$ which results $D(x - \dddot{x}) = 0$. We can use (21) in JS-RIP as follows
\[
(1 - \delta_{2K})||x - \dddot{x}||_2^2 \leq ||D(x - \dddot{x})||_2^2 = 0 \tag{22}
\]

Then we can see that $||x - \dddot{x}||_2^2 = 0$ since $\delta_{2K} < 1$ which concludes the proof. \hfill $\square$

Fig. 1. The placement of the antennas in the array for a single realization. Dot: the positions of the fixed antennas, Circle: the positions of the randomly changing antennas.

6. SIMULATIONS

In this section, the proposed method is evaluated by several experiments. Throughout the simulations, we consider three sources located at $32.375^\circ, 50.714^\circ$ and $75.215^\circ$ in azimuth plane. Multi-resolution grid refinement [19, Sec. 6] is used to resolve off-grid targets. $\Delta \theta = 1$ degree is selected for coarse resolution for the dictionary matrix $A$. Note that $A$ is used for conventional CS algorithms while $D$ is used for the proposed method. The regularization parameter, $\mu$ is selected as described in [20]. 100 Monte-Carlo trials are run for each experiment. In each trial, the positions of the antennas in the array are updated. There are $M = 16$ antennas positioned randomly in $xy$-plane. The positions of the antennas $(x_m, y_m)$ are uniformly distributed as $x_m \in [-\lambda, \lambda]$ and $y_m \in [-\lambda, \lambda]$ for $m = 1, \ldots, M$. In order to avoid from a dense antenna positioning, each antenna is assumed to be placed randomly in a box in $xy$-plane. The $xy$-plane is divided into $M = 16$ boxes as shown in Fig. 1 where a single realization for the positions of the antennas is presented. The position of the first two antennas are fixed with a half wavelength spacing, namely $(x_1, y_1) = (\frac{-\lambda}{2}, \frac{\lambda}{2})$ and $(x_2, y_2) = (\frac{\lambda}{2}, \frac{-\lambda}{2})$. While this is not required for the proposed approaches, this selection is used to avoid spatial aliasing. The other antennas are randomly distributed such that there is a single antenna in each box.

In order to generate the MC matrix, a similar approach is followed as in [2] and [12] where MC coefficients are selected randomly based on the distance between antennas. Note that it is assumed that there are at most $M$ unknown coupling coefficients so that the problem in (17) is well-posed. The coupling coefficients for each antenna pair with $d_{m} < \lambda$ are calculated as a function of the distance between antennas where $d_{m}$ is the inter-element distance for the $m^{th}$ antenna pair and $m = 1, \ldots, M$. The coupling coefficients corresponding to the antenna pairs with $d_{m} \geq \lambda$ are selected as zero [15]. The gain and the phase terms of coefficients are selected as $c_m = G_m \exp(jc_m)$ where $G_m \sim \mathcal{N}\left(\frac{\lambda}{2}, \left(\frac{\lambda}{2\sigma}\right)^2\right)$ and $c_m \sim \mathcal{N}\left(0, \left(\frac{\lambda}{2\sigma}\right)^2\right)$ for $m = 2, \ldots, M$ respectively. Note that $G_m^2 = 1$ and $c_m = 0$ for $m = 1$. $d = [d_{2}, \ldots, d_{M}] = \frac{d}{|\Delta d|}$ is a $(M - 1) \times 1$ vector composed of the normalized distances. Hence, coupling coefficient, $c_m$, will be close to 1 in magnitude as $|d_m| \rightarrow 0$ and it will be close to 0 as $|d_m| \rightarrow 1$.

In the simulations, the proposed method is compared with Ba-
sis Pursuit De-noising (BPDN) [20], Orthogonal Matching Pursuit (OMP) [21] and the MUSIC algorithm [1] with and without known MC as well as the Cramer-Rao lower Bound (CRB) [2].

In Fig. 2, DOA estimation performance of the algorithms is presented for a single snapshot. As it is seen, The proposed method asymptotically follows CRB while the other algorithms do not give accurate estimation results. The MUSIC algorithm also fails due to rank-deficiency. While BPDN and OMP are very effective algorithms for MC-free measurements, their performances are not satisfactory when there is MC.

In Fig. 3, root mean square error (RMSE) for the gain term of the MC coefficients is presented. The proposed method has significant performance improvement compared to the alternative methods for gain estimates. In Fig 4, estimation performance for the phase of the MC coefficients is given. While the proposed method has again performed better, the margin between the proposed method and CRB is larger in comparison.

In order to evaluate the proposed approach with a uniform array structure, a UCA is considered for $M = 16$ antennas with $\lambda/2$ element-spacing. Note that there are $\bar{M} = 9$ coupling coefficients for circulant MC matrix of UCA. The results are shown in Fig. 5. As it is seen, the proposed method closely follows the CRB while OMP and BPDN cannot achieve accurate results.

7. CONCLUSIONS

In this paper, joint-sparse recovery algorithm for a single snapshot is proposed for the estimation of off-grid source DOA angles and mutual coupling coefficients using randomly placed array structures. The proposed method is compared with both conventional compressed sensing techniques and subspace algorithms as well as Cramer-Rao lower bound. It is shown that conventional CS methods cannot perform well due to the effect of MC. The subspace methods also fail in this scenario due to rank-deficiency. The proposed method is shown to perform well for a variety of scenarios in Monte-Carlo experiments.
8. REFERENCES


