SIMPLIFIED MATRIX POLYNOMIAL-AIDED BLOCK DIAGONALIZATION PRECODING FOR MASSIVE MIMO SYSTEMS

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ABSTRACT

In the downlink of Massive Multiple-Input-Multiple-Output (MIMO) systems, the high computational cost of precoding is a major challenge for real-time data transmission. In this paper, we propose a simplified matrix polynomial-aided block diagonalization (SMP-BD) precoding scheme to simplify the conventional block diagonalization (BD) type precoding schemes for users with multiple antennas. By replacing the singular value decomposition (SVD) operation with a matrix inversion, which is then approximated by matrix polynomials with optimized coefficients, SMP-BD is shown to be hardware-efficient, reduce the transmission overhead and obtain high performance. The optimized coefficients can be calculated offline and the convergence of the matrix polynomials is guaranteed. Simulations show that SMP-BD with polynomial order $L = 3$ performs close to previously reported algorithms based on matrix inversions, while being much simpler.

Index Terms— Massive MIMO, block diagonalization (BD) precoding, minimum mean square error (MMSE), matrix inversion, matrix polynomials, optimal coefficients.

1. INTRODUCTION

Massive Multiple-Input-Multiple-Output (MIMO) systems are an enhanced version of MIMO systems, which are equipped with a much larger number of antennas at the base station (BS), that have drawn great research interest recently [1–3]. Despite the attractive advantages of Massive MIMO systems that include high capacity [1], improved link reliability, better coverage [2] and excellent energy efficiency [4], for practical implementations there are still many challenges such as channel estimation [5], hardware imperfections [3, 6, 7] and low complexity signal processing including precoding and decoding [2].

Most previous works on Massive MIMO downlink precoding considered single-antenna users, e.g., matched filter (MF), zero forcing (ZF) and linear minimum mean squared error (MMSE) precoders (see [8] and references therein). However, by employing multiple antennas at the user equipment (UE), the quality of service (QoS) can be dramatically improved. In fact, the long-term evolution (LTE) and LTE-Advanced (LTE-A) standards can support UEs with multiple antennas. For multiple-antenna users, block diagonalization (BD) type precoding has been widely used for parallel transmission of multiple data streams [9–14]. In [9, 10], the authors have extended ZF precoding to users with multiple antennas and proposed BD algorithms, which employ two precoders to eliminate inter- and intra-user interference, respectively. The work in [11] has devised regularized BD (RBD) algorithm by taking into account both the inter-user interference and the noise at the receiver, which improves the performance of BD in low signal-to-noise ratio (SNR) region. The main drawback of BD and RBD is the complexity which mainly comes from the involved singular value decomposition (SVD) operations. In order to reduce the computational complexity, [12] replaced the SVD with generalized MMSE channel inversion (GMI), which is further simplified in [14] as simplified GMI (S-GMI). However, the calculation of a matrix inversion is still not hardware-efficient for Massive MIMO systems.

In this paper, we propose a simplified matrix polynomial-aided BD (SMP-BD) algorithm, which replaces the SVD used in BD or the matrix inversion used in GMI with a matrix polynomial expansion. In contrast to related works on matrix inversion simplification based on matrix polynomials in [15–17], we introduce an auxiliary parameter $\beta$ in addition to the polynomial coefficients. These parameters are optimized by solving an optimization problem and the convergence of the polynomials is guaranteed. With this matrix polynomial expansion, the computational complexity is greatly reduced and the hardware implementation can be facilitated. Moreover, we also simplify the design of the second part of the S-GMI precoder based on the MMSE criterion. Simulations show that the proposed matrix inversion scheme outperforms [15] and [16] significantly. Numerical results indicate that with small order of polynomials ($L = 3$), SMP-BD performs close to S-GMI, which involves a full matrix inversion.

2. SYSTEM MODEL AND BD PRECODING

2.1. System Model

Consider the downlink of a Massive MIMO system with an $N$-antenna BS and $K$ users, where there are $M_k$ antennas at the $k$-th user. Define $M = \sum_{k=1}^{K} M_k$ as the total number of antennas at all the users. The received signal $y_k \in \mathbb{C}^{M_k}$ at the $k$-th user is given by

$$y_k = H_k x + n_k,$$  

(1)
where $H_k \sim \mathcal{CN}(0, I_N)$ is the Rayleigh fading channel matrix of the $k$-th user whose entries have zero mean, unit variance and are independent and identically distributed (iid); $n_k \sim \mathcal{CN}(0, \sigma_n^2 I_{M_k})$ is the noise vector at the receiver, $x \in \mathbb{C}^N$ is the transmit signal vector after precoding given by $x = Ps$, where $P$ and $s$ are the combined precoding matrix and transmit signal vector for all the users, respectively.

The precoding matrix and the transmit signal vector are given by

$$P = [P_1, \ldots, P_K], \quad s = [s_1^T, \ldots, s_K^T]^T,$$

in which $P_k \in \mathbb{C}^{N \times L_k}$ and $s_k \in \mathbb{C}^{L_k}$ are the precoder and the transmit signal vector for the $k$-th user, respectively, and $L_k$ is the number of data streams of user $k$. We assume that the transmit signal vectors for different users are independent with unit energy, i.e., $\forall k \neq j$, $E[s_k s_j^H] = 0$, and $E[s_k s_k^H] = I_{L_k}$. Denoting $H = [H_1^T, \ldots, H_K^T]^T$, $y = [y_1^T, \ldots, y_K^T]^T$ and $n = [n_1^T, \ldots, n_K^T]^T$, we have the received data vector for all the users given by

$$y = HPs + n. \quad (2)$$

### 2.2. Introduction to BD Precoding

For users with multiple antennas, one well known precoding scheme is BD [9, 10]. The BD precoding matrix of the $k$-th user is composed of four parts: $P_k = \lambda P_{\text{BD}} L_k \Gamma_k$, in which $P_{\text{BD}} \in \mathbb{C}^{N \times L_k}$ (the value of $D_k$ is related to the way of choosing $P_{\text{BD}}$ and $P_k$); $\Gamma_k \in \mathbb{C}^{L_k \times L_k}$ is the diagonal power loading matrix, i.e., $\text{ETr}(P_k^* P_k) = \text{Tr}(P_k)$. Let us exclude the $k$-th user’s channel matrix and define $H_{-k} \in \mathbb{C}^{M_k \times N}$ as $H_{-k} = [H_1^T, \ldots, H_{k-1}^T, H_{k+1}^T, \ldots, H_K^T]^T$, where $M_k = M - M_k$.

In order to eliminate inter-user interference, $P_{\text{BD}}$ is chosen to be in the null space of $H_{-k}$, i.e., $H_{-k} P_{\text{BD}} = 0$. One way to solve this problem is by applying SVD to $H_{-k}$ and choosing $P_{\text{BD}}$ as the right singular vectors corresponding to the zero singular values [9, 10]. Denote the SVD of $H_{-k}$ as $H_{-k} = U_{-k} \Sigma_{-k} V_{-k}^H$. The second precoding matrix and the corresponding receive filter matrix are obtained, respectively, as $P_{\text{BD}} = V_{-k}$, $C_k = U_{-k}$.

To calculate the BD precoder, $K$ times SVD of $M_k \times N$ matrices and $K$ times SVD of $M_k \times D_k$ matrices are required. Because SVD is computationally very costly and not hardware friendly, the complexity of BD impedes its implementation in Massive MIMO systems.

### 3. Proposed Simplified Matrix Polynomial-Aided Block Diagonalization Precoding

In this section, we propose a simplified matrix polynomial-aided block diagonalization precoding (SMP-BD) scheme, which simplifies the traditional BD precoder based on the following two ideas:

1. To replace the SVD while calculating $P_{\text{BD}}$ with a matrix inversion approximated by matrix polynomials, the coefficients of which are optimized;
2. To replace the SVD involved in the calculation of $P_{\text{BD}}$ with a matrix inversion, thus reducing the complexity as well as the cost of system resources. For simplicity, in the following we assume full data rate transmission and equal power allocation for all the users, which means $D_k = L_k = M_k$ and $\Gamma_k = I_{L_k}$. We also provide convergence results based on random matrix theory, which only hold for channels whose entries have zero mean, unit variance and are iid.

#### 3.1. Calculation of $P_{\text{SMP}}$

Since when $N$ and $M$ are large it is computationally expensive to calculate an SVD $K$ times for BD, we adopt an alternative approach based on S-GMI [14]. By applying the MMSE matrix inversion to $H$, we have

$$H^H (HH^H + \rho I_M)^{-1} = [H_1^H, \ldots, H_K^H]. \quad (4)$$

where $H_k^H \in \mathbb{C}^{N \times M_k}$ and $\rho = \frac{\sigma_n^2}{\text{Tr}(H H^H)}$. [18]

Applying a QR decomposition to $H_k^H$ yields $H_k^H = F_k R_k$, where $F_k \in \mathbb{C}^{N \times M_k}$ is an orthogonal matrix and $R_k \in \mathbb{C}^{M_k \times M_k}$ is an upper triangular matrix. The first precoder is thus chosen as $P_1 = F_1$ [12].

For large values of $M$, the matrix inversion in (4) has large computational complexity and is not efficient in hardware implementation [17]. In general, multiplications are preferable to divisions in hardware. Therefore, we propose to approximate the matrix inversion by matrix polynomials based on Lemma 1 and optimize the polynomial coefficients.

**Lemma 1 (Neumann Series, [19]).** If a matrix $Z \in \mathbb{C}^{M \times M}$ has the property that $\lim_{t \to \infty} (I_M - Z)^t \to 0$, then $Z$ is nonsingular and its inverse may be expressed by a Neumann series as $Z^{-1} = \sum_{n=0}^{\infty} (I_M - Z)^n$.

Let us denote $A \triangleq H H^H + \rho I_M$ and $B \triangleq I_M - \beta A$ where $\beta$ is a design parameter to fulfill Lemma 1. Then the following approximation is achieved

$$A^{-1} = \beta \sum_{n=0}^{\infty} B^n \approx \beta \sum_{n=0}^{L} \alpha_n B^n, \quad (5)$$

where $L \geq 0$ is the order of matrix polynomials and $\alpha_n$ is the polynomial coefficient to be optimized.

Denote $\alpha \triangleq [\alpha_0, \ldots, \alpha_L]^T$. To select the optimal $\beta$ and $\alpha$, consider the following optimization problem:

$$\{\beta^*, \alpha^*\} = \arg \min_{\beta \in \mathbb{C}, \alpha \in \mathbb{C}^{L+1}} \mathbb{E} [A^{-1} - \beta \sum_{n=0}^{L} \alpha_n B(\beta)^n]^2 \quad \text{s.t. } \rho_1(B) < 1, \quad (6)$$

where $\| \cdot \|_F$ is the Frobenius norm and $\rho_1(\cdot)$ denotes the spectrum radius of a matrix.

The above optimization problem is generally non-convex and the globally optimal solution is difficult to find. Therefore, we derive the sub-optimal solution to (6) by first choosing an appropriate $\beta$ and then optimizing $\alpha$. 


3.1.1. Choices of $\beta$

$\beta$ is chosen to ensure the spectrum radius of $B$ is smaller than 1 so as to fulfill Lemma 1. According to [20], in massive MIMO systems when $M$ and $N$ becomes large with a fixed ratio $\eta = \frac{M}{N}$, the eigenvalue distribution of $\frac{1}{N} HH^H$ converges to

$$f_{\frac{1}{N} HH^H}(z) = (1 - \eta^{-1})^+ \delta(z) + (2\pi\eta)^{-1}(z-a)^+\sqrt{(b-z)^2},$$  (7)

where $a = (1 - \sqrt{\eta})^2$ and $b = (1 + \sqrt{\eta})^2$.

Consider $\eta < 1$ which is the prevailing scenario. When $M$ and $N$ are large, we have

$$\rho_i(B) \approx \max\{|1 - \beta(\rho + Nb)|, |1 - \beta(\rho + Na)|\}. \quad (8)$$

In order to facilitate fast convergence, the optimal $\beta$ should minimize $\rho_i(B)$. Therefore, $\beta$ is chosen by solving the following optimization problem:

$$\beta^* = \arg \min_{s.t. \rho_i(B) < 1} \rho_i(B). \quad (9)$$

Proposition 1. The optimal $\beta$ achieved by solving (9) is given by

$$\beta^* = \frac{2}{2\rho + N_a + N_b}. \quad (10)$$

Proof. The proposition is proved by dividing the set of feasible $\beta$ into three cases and comparing the spectrum radius of $B$. The details are omitted due to page limits.

From Proposition 1, the following observation on convergence is achieved.

Observation 1. The optimal choice of $\beta$ in Proposition 1 guarantees the convergence of the matrix polynomials.

Proof. The proof is straightforward since $\rho_i(B) = \frac{N_a - N_b}{2\rho + N_a + N_b} < 1$.

3.1.2. Optimization of Polynomial Coefficients

With the value of $\beta$ in hand, (6) is simplified to

$$\alpha^* = \arg \min_{\alpha \in \mathbb{C}^{L+1}} \mathbb{E}\left\| A^{-1} - \beta \sum_{n=0}^{L} \alpha_n B^n \right\|^2_F. \quad (11)$$

The above optimization problem is convex in $\alpha$. The solution to (10) is given by Proposition 2.

Proposition 2. The optimal $\alpha^*$ in (10) is given by

$$\alpha^* = G^{-1} w, \quad (12)$$

where $G$ and $w$ are given by

$$G = \begin{bmatrix} \mathbb{E}Tr B^0 & \cdots & \mathbb{E}Tr B^L \\ \vdots & \ddots & \vdots \\ \mathbb{E}Tr B^L & \cdots & \mathbb{E}Tr B^{L+L} \end{bmatrix}, \quad (13)$$

$$w = \mathbb{E}Tr\{\beta^{-1} A^{-1} B^0\}, \ldots, \mathbb{E}Tr\{\beta^{-1} A^{-1} B^L\}^T.$$

in which we have

$$\mathbb{E}Tr B^t = M \int (1 - \beta z)^t f_A(z) dz, \quad (14)$$

where $f_A(z)$ is the empirical eigenvalue distribution of $A$, which is obtained from (7) when $M, N \to \infty$ and is given by

$$f_A(z) = \frac{1}{N} \mathbb{E}_{\frac{1}{N} HH^H}(\frac{2 - \beta}{N} z). \quad (15)$$

Proof. The proof follows by setting the first-order derivative to zero and solving an equation. The details are omitted due to page limits.

Note that $\alpha$ can be calculated offline. Therefore, it does not affect real-time applications. In (12), $G$ is invertible in most cases because this matrix has full rank. Although (15) is valid for $M, N \to \infty$, the result in Proposition 2 is very accurate for small values of $M$ and $N$.

3.2. Calculation of $P_{k2}$

Once $P_{k1}$ is obtained, the effective channel matrix can be obtained as in (3), which is used to calculate $P_{k2}$ and $G_k$. In standard BD methods, user $k$ obtains $G_k$ by: 1) feed-forwarding from the BS; or 2) estimating $H_{ke}$ and calculating $G_k$ itself. However, either case will need additional system resources.

In this work, we propose to compute $P_{k2}$ according to the MMSE criterion with positive auxiliary parameter $t$:

$$P_{k2} = \arg \min \mathbb{E}\|H_{ke} P_{k2} s_k + e_k + n_k - ts_k\|^2_F, \quad (16)$$

where $e_k$ is the other users’ interference and is treated as noise with covariance of $\delta_k^2 I_{M_k}$ for simplicity. Solving (16) gives

$$P_{k2}^* = H_{ke}^H (H_{ke} H_{ke}^H + \rho_2 I_{M_k})^{-1}, \quad (17)$$

where $\rho_2 = M(\delta_k^2 + \sigma^2)/PT$.

In fact, it is not easy to derive the optimal $\delta_k^2$, which is left for future work. Here, we simply select the sub-optimal $\delta_k^2 = 0$.

The reason is that in (16), the exact expression for $e_k$ is given by $e_k = H_{ke} \sum_{j \neq k} P_j s_j$, the covariance of which is $\mathbb{E}\{e_k e_k^H\} = H_{ke} \sum_{j \neq k} P_j P_j^H H_{ke}^H$. Since $H_{ke} P_j \approx 0$ especially when $\rho$ is small, it is reasonable to set $\delta_k^2$ to zero.

The benefits of (17) are twofold: 1) The SVD operation is replaced by a matrix inversion, and thus the complexity is reduced. 2) No receive filter is needed. Therefore, no feed-forwarding or channel estimation is required, which helps reduce the transmission overhead.

4. PERFORMANCE ANALYSIS

In this section, we present a performance analysis for the proposed SMP-BD algorithm in terms of sum data rates and computational complexity.

4.1. Sum Data Rates

Denote $R_k$ as the data rate of the $k$-th user. The sum rate is then given by $R_{sum} = \sum_k R_k$.

Let $G_k$ be the receive filter of the $k$-th user, and then multiplying the received signal vector by $G_k$ yields

$$d_k = G_k y_k = G_k H_k P_k s_k + G_k H_k P_{-k} s_{-k} + G_k n_k \triangleq Q_k s_k + Q_{-k} s_{-k} + G_k n_k,$$
where
\[ P_{-k} = [P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_N], \]
\[ s_{-k} = [s_1^T, \ldots, s_{k-1}^T, s_{k+1}^T, \ldots, s_N^T], \]
and \( Q_k = G_k H_k P_k \quad \text{and} \quad Q_{-k} = G_k H_k P_{-k}. \) Assuming Gaussian signaling is used, the data rate of the \( k \)-th user is thus given by
\[
R_k = \log_2 \left( \frac{\det(Q_k Q_k^H + Q_{-k} Q_{-k}^H + \sigma_n^2 G_k G_k^H)}{\det(Q_{-k} Q_{-k}^H + \sigma_n^2 G_k G_k^H)} \right).
\]

4.2. Complexity

In order to show the complexity of the proposed SMP-BD algorithm, we count the number of different matrix operations, i.e., SVD, QR decomposition, matrix inversion (MI), matrix multiplication (MM), which dominate the computational complexity. The complexity of the proposed SMP-BD algorithm mainly comes from the calculation of \( P_{k1} \). Since \( \beta \) and \( \alpha \) can be calculated offline. Therefore, we omitted them here. The complexity comparison of BD [9, 10], S-GMI [14] and SMP-BD is summarized in Table 1.

<table>
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<tr>
<th>Method</th>
<th>SVD</th>
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<tr>
<td>S-GMI</td>
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<tr>
<td>SMP-BD</td>
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<td>K</td>
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<td>( L + 1 )</td>
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Table 1. Complexity Comparison for the First Precoder.

In the simulations, \( L = 3 \) is good enough in most cases. Therefore, with two more additional matrix multiplications than S-GMI, SMP-BD can avoid the matrix inversion and maintain almost the same performance.

5. NUMERICAL RESULTS

In this section, we carry out Monte-Carlo simulations to show the performance of the proposed SMP-BD algorithm. The performance of SMP-BD is compared with that of S-GMI in [14], because: 1) S-GMI has similar performance as RBD in [11] and is shown to outperform BD in [9, 10]; 2) SMP-BD is a simplified approach based on S-GMI. In all the simulations, \( N = 500, K = 50, M_1 = \ldots = M_K = 2 \). The SNR is defined as \( \frac{\|F\|}{\eta} \). The bit error rate (BER) is obtained based on Quadrature Phase Shift Keying (QPSK) modulation.

Fig. 1 shows the comparison of the approximation error of matrix polynomials between the proposed method and those in [15] and [16]. The deviation \( \tau \) is defined as \( \tau = \mathbb{E} \left( \frac{\|A^{-1} - A_{\text{opt}}^{-1}\|}{\|A_{\text{opt}}^{-1}\|} \right) \), which is used to show the relative approximation error of \( A_{\text{opt}} \) obtained by the matrix polynomials. As can be seen in Fig. 1, as the order of the matrix polynomials increases, \( \tau \) decreases exponentially, which demonstrates the fast convergence of the proposed scheme. It can be shown that proposed scheme significantly outperforms existing methods in [15] and [16].

Fig. 2 shows the comparison between S-GMI in [14] and the proposed SMP-BD with \( L = 1, 2, 3 \) in terms of sum rates and BER, respectively. It is shown that the performance of SMP-BD improves as \( L \) increases. For \( L = 2 \), SMP-BD is able to achieve over 80% of the sum rates of S-GMI. When \( L = 3 \), SMP-BD almost performs the same as S-GMI.

Although the performance of SMP-BD is slightly worse than S-GMI, it does not need to compute a matrix inversion, and thus is hardware-friendly. Moreover, with the simplified method of calculating \( P_{k2} \), the system overhead is greatly reduced.

6. CONCLUSIONS

In this paper, we have proposed the SMP-BD algorithm to simplify the conventional BD type precoders which have high computational complexity and are not suitable for Massive MIMO systems. We first used matrix polynomials with optimized coefficients to simplify the first precoder, and then reduced the overhead in calculating the second precoder based on the MMSE criterion. The proposed SMP-BD is hardware-efficient and performs close to S-GMI for \( L = 3 \) with much simplified design.

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8. REFERENCES


